

MIXED INITIAL-BOUNDARY VALUE PROBLEM FOR THE THREE-DIMENSIONAL NAVIER-STOKES EQUATIONS IN POLYHEDRAL DOMAINS

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ABSTRACT. We study a mixed initial-boundary value problem for the Navier-Stokes equations, where the Dirichlet, Neumann and slip boundary conditions are prescribed on the faces of a three-dimensional polyhedral domain. We prove the existence, uniqueness and smoothness of the solution on a time interval $(0, T^*)$, where $0 < T^* \leq T$.

1. INTRODUCTION

1.1. Preliminaries. We consider a mixed initial-boundary value problem for the Navier-Stokes equations in a three-dimensional domain $\Omega \subset \mathbb{R}^3$ of polyhedral type with a boundary $\partial\Omega$. The domain Ω represents e.g. a channel filled up by a moving fluid. $\partial\Omega$ consists of nonintersecting pieces Γ_D , Γ_G and Γ_N , $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_G} \cup \overline{\Gamma_N}$. $\Gamma_D = \bigcup_{j \in \mathcal{J}_1} \Gamma_j$ represents solid walls, $\Gamma_G = \bigcup_{j \in \mathcal{J}_2} \Gamma_j$ denotes uncovered fluid surfaces and $\Gamma_N = \bigcup_{j \in \mathcal{J}_3} \Gamma_j$ denotes the artificial part of the boundary such as the exit (or a free surface), $\mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3 = \{1, \dots, n\}$ and $\Gamma_i \cap \Gamma_j = \emptyset$ iff $i \neq j$, $i, j \in \{1, \dots, n\}$.

We study the existence and uniqueness of the solution \mathbf{u} to the Navier-Stokes flows on $(0, T)$, $T > 0$, in Ω under the following boundary conditions:

$$\begin{aligned} (1) \quad & \mathbf{u} = \mathbf{0} && \text{on } \Gamma_D \times (0, T), \\ (2) \quad & \mathbf{u} \cdot \mathbf{n} = 0, \quad [\nabla \mathbf{u} + (\nabla \mathbf{u})^\top] \mathbf{n} \cdot \boldsymbol{\tau} = 0 && \text{on } \Gamma_G \times (0, T), \\ (3) \quad & -\mathcal{P} \mathbf{n} + \nu [\nabla \mathbf{u} + (\nabla \mathbf{u})^\top] \mathbf{n} = \mathbf{0} && \text{on } \Gamma_N \times (0, T). \end{aligned}$$

In 1–3 $\mathbf{u} = (u_1, u_2, u_3)$ and \mathcal{P} denote the unknown velocity and pressure, respectively. Further, ν denotes the viscosity of the fluid. $\mathbf{n} = (n_1, n_2, n_3)$ and $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ are unit normal and tangent vectors, respectively, to $\partial\Omega$.

1.2. The domain. It is well known that the regularity results for solutions of elliptic problems in domains with edges or with the mixed boundary conditions are closely related to the properties of the boundary of the domain. Hence we specify several attributes of the domain Ω , which will be used later. We assume that

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- (i) Γ_i (the faces of Ω), $i = 1, \dots, n$, are open two-dimensional manifolds of class C^∞ ;
- (ii) the boundary $\partial\Omega$ consists of smooth faces Γ_i (defined above) and smooth (of class C^∞) nonintersecting curves \mathcal{M}_k (the edges), $k = 1, \dots, m$, vertices on $\partial\Omega$ are excluded;
- (iii) for every $A \in \mathcal{M}_k$, $k = 1, \dots, m$, there exists a neighborhood \mathcal{U}_A and a diffeomorphic mapping κ_A which maps $\Omega \cap \mathcal{U}_A$ onto $\mathcal{D}_A \cap B_A$, where \mathcal{D}_A is a dihedron of the form

$$\{[x_1, x_2, x_3] \in \mathbb{R}^3; 0 < r < \infty, -\omega_A/2 < \varphi < \omega_A/2, x_3 \in \mathbb{R}\},$$

$\omega_A > 0$ denotes the angle at the edge \mathcal{M}_k , $A \in \mathcal{M}_k$, and B_A is the unit ball (r, φ denote the polar coordinates in the (x_1, x_2) -plane);

- (iv) $\Gamma_i \in \Gamma_D$, i.e. $i \in \mathcal{J}_1$, forms at least one of the adjoining faces of every edge \mathcal{M}_k , $k = 1, \dots, m$;
- (v) $\begin{cases} \text{for every } A \in \mathcal{M}_k, \mathcal{M}_k \subset \overline{\Gamma_D} \cap \overline{\Gamma_D}, & \text{we have } \omega_A < \pi, \\ \text{for every } A \in \mathcal{M}_k, \mathcal{M}_k \subset \overline{\Gamma_D} \cap \overline{\Gamma_G}, & \text{we have } \omega_A < (3/4)\pi, \\ \text{for every } A \in \mathcal{M}_k, \mathcal{M}_k \subset \overline{\Gamma_D} \cap \overline{\Gamma_N}, & \text{we have } \omega_A < (1/4)\pi. \end{cases}$

1.3. Basic notation and some function spaces. Vector functions and operators acting on vector functions are denoted by boldface letters. Unless specified otherwise, we use Einstein's summation convention for indices running from 1 to 3.

For an arbitrary $r \in [1, +\infty]$, $L^r(\Omega)$ denotes the usual Lebesgue space equipped with the norm $\|\cdot\|_{L^r(\Omega)}$, and $W^{k,p}(\Omega)$, $k \geq 0$ (k need not to be an integer, see [12]), $1 \leq p < \infty$, denotes the usual Sobolev space with the norm $\|\cdot\|_{W^{k,p}(\Omega)}$.

Let

$$E := \{\mathbf{u} \in C^\infty(\overline{\Omega})^3; \operatorname{div} \mathbf{u} = 0, \operatorname{supp} \mathbf{u} \cap \Gamma_D = \emptyset, \operatorname{supp} \mathbf{u} \cdot \mathbf{n} \cap \Gamma_G = \emptyset\}$$

and $V^{k,p}$ be a closure of E in the norm of $W^{k,p}(\Omega)^3$, $k \geq 0$ (k need not be an integer) and $1 \leq p < \infty$. Then $V^{k,p}$ is a Banach space with the norm of the space $W^{k,p}(\Omega)^3$. For simplicity, we denote $V^{1,2}$ and $V^{0,2}$, respectively, as V and H . Note, that V and H , respectively, are Hilbert spaces with scalar products

$$(4) \quad ((\mathbf{u}, \mathbf{v})) = 2 \int_{\Omega} \varepsilon_{ij}(\mathbf{u}) \varepsilon_{ij}(\mathbf{v}) \, d\Omega$$

and

$$(\mathbf{u}, \mathbf{v}) = \int_{\Omega} u_i v_i \, d\Omega$$

and they are closed subspaces of spaces $W^{1,2}(\Omega)^3$ and $L^2(\Omega)^3$, respectively. In 4 $e_{ij}(\mathbf{u})$ denotes the matrix with the components

$$e_{ij}(\mathbf{u}) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

Further, define the space

$$(5) \quad \mathcal{D} := \{\mathbf{u} \mid \mathbf{f} \in H, ((\mathbf{u}, \mathbf{v})) = (\mathbf{f}, \mathbf{v}) \text{ for all } \mathbf{v} \in V\}$$

equipped with the norm

$$\|\mathbf{u}\|_{\mathcal{D}} := \|\mathbf{f}\|_H,$$

where \mathbf{u} and \mathbf{f} are corresponding functions via 5.

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in W^{1,2}(\Omega)^3$. We will use the notation

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} \, d\Omega.$$

Throughout the paper, we will always use positive constants c, c_1, c_2, \dots , which are not specified and which may differ from line to line.

2. FORMULATION OF THE PROBLEM

Let $T \in (0, \infty)$, $Q = \Omega \times (0, T)$. The classical formulation of our problem is as follows:

$$\begin{aligned} (6) \quad & \mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla \mathcal{P} = \mathbf{f} && \text{in } Q, \\ (7) \quad & \operatorname{div} \mathbf{u} = 0 && \text{in } Q, \\ (8) \quad & \mathbf{u} = \mathbf{0} && \text{on } \Gamma_D \times (0, T), \\ (9) \quad & \mathbf{u} \cdot \mathbf{n} = 0, \quad [\nabla \mathbf{u} + (\nabla \mathbf{u})^\top] \mathbf{n} \cdot \boldsymbol{\tau} = 0 && \text{on } \Gamma_G \times (0, T), \\ (10) \quad & -\mathcal{P} \mathbf{n} + \nu [\nabla \mathbf{u} + (\nabla \mathbf{u})^\top] \mathbf{n} = \mathbf{0} && \text{on } \Gamma_N \times (0, T), \\ (11) \quad & \mathbf{u}(0) = \mathbf{u}_0 && \text{in } \Omega. \end{aligned}$$

Here \mathbf{f} is a body force and \mathbf{u}_0 describes an initial velocity. We assume that functions $\mathbf{u}, \mathcal{P}, \mathbf{f}$ and \mathbf{u}_0 are smooth enough and the compatibility conditions $\mathbf{u}_0 = \mathbf{0}$ on Γ_D and $\mathbf{u}_0 \cdot \mathbf{n} = 0$ on Γ_G hold. For simplicity we suppose that $\nu = 1$ throughout the paper.

We can formulate our problem:

Suppose that $\mathbf{f} \in L^2(0, T; H)$ and $\mathbf{u}_0 \in \mathcal{D}^1$. Find $\mathbf{u} \in L^2(0, T; \mathcal{D}) \cap L^\infty(0, T; V)$, $\mathbf{u}_t \in L^2(0, T; H)$ such that

$$(12) \quad (\mathbf{u}_t, \mathbf{v}) + ((\mathbf{u}, \mathbf{v})) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})$$

for every $\mathbf{v} \in V$ and for almost every $t \in (0, T)$ and

$$(13) \quad \mathbf{u}(0) = \mathbf{u}_0.$$

The main difficulties of problem 12–13 consist in the fact that, because of the artificial boundary condition 10, we cannot prove that $b(\mathbf{u}, \mathbf{u}, \mathbf{u}) = 0$. Consequently, we are not able to show that the kinetic energy of the fluid is controlled by the data of the problem and the solutions of 12–13 need not satisfy the energy inequality. This is due to the fact that some uncontrolled “backward flow” can take place at the open parts Γ_N of the domain Ω and one is not able to prove global (in time) existence results. In [5]–[7], Kračmar and Neustupa prescribed an additional condition on the output (which bounds the kinetic energy of the backward flow) and formulated steady and evolutionary Navier–Stokes problems by means of appropriate variational inequalities. In [11], Kučera and Skalák prove the local-in-time existence and uniqueness of a “weak” solution of the non-steady Navier–Stokes problem with boundary condition 10 on the part of the boundary $\partial\Omega$, such that

$$(14) \quad \mathbf{u}_t \in L^2(0, T^*; V), \quad \mathbf{u}_{tt} \in L^2(0, T^*; V^*), \quad 0 < T^* \leq T,$$

¹The requirement $\mathbf{u}_0 \in \mathcal{D}$ represents an implicit compatibility condition imposed on the initial data.

under some smoothness restrictions on \mathbf{u}_0 and \mathcal{P} . In [10], Kučera supposed that the problem is solvable in suitable function class with some given data (the initial velocity and the right hand side). The author proved that there exists a unique solution for data which are small perturbations of the previous ones.

In [3], Beneš and Kučera proved local existence of solutions to the Navier–Stokes system with the so called “do nothing” boundary condition

$$-\mathcal{P}\mathbf{n} + \frac{\partial \mathbf{u}}{\partial \mathbf{n}} = \mathbf{0} \quad \text{on } \Gamma_N \times (0, T)$$

for sufficiently smooth data in a stronger (spatially) sense than 14 without higher regularity with respect to time. However, the authors excluded the boundary condition 9 and proved the local existence solution solely in two dimensions.

In the present paper, we shall prove local existence and uniqueness solution to 6–11 such that i.a. $\mathbf{u} \in L^2(0, T^*; \mathcal{D})$, $\mathcal{D} \hookrightarrow W^{2,2}(\Omega)^3$, which is strong in the sense that the solutions possess second spatial derivatives. The key embedding $\mathcal{D} \hookrightarrow W^{2,2}(\Omega)^3$ is a consequence of assumptions setting on the domain Ω and the regularity theory for the steady Stokes system in non-smooth domains, see [19, Corollary 4.2] and [15, 16].

In next Section 3 we present some auxiliary results needed in the proof of the main result stated and proved in Section 4.

3. AUXILIARY RESULTS

Theorem 3.1 (Linearized problem). *Let $\mathbf{f} \in L^2(0, T; H)$. Then there exists unique function $\mathbf{u} \in L^2(0, T; \mathcal{D}) \cap L^\infty(0, T; V)$, $\mathbf{u}_t \in L^2(0, T; H)$, such that*

$$(15) \quad (\mathbf{u}_t, \mathbf{v}) + ((\mathbf{u}, \mathbf{v})) = (\mathbf{f}, \mathbf{v})$$

holds for every $\mathbf{v} \in V$ and for almost every $t \in (0, T)$ and

$$(16) \quad \mathbf{u}(0) = \mathbf{0}.$$

Moreover

$$(17) \quad \|\mathbf{u}_t\|_{L^2(0, T; H)} + \|\mathbf{u}\|_{L^2(0, T; \mathcal{D})} + \|\mathbf{u}\|_{L^\infty(0, T; V)} \leq c \|\mathbf{f}\|_{L^2(0, T; H)},$$

where $c = c(\Omega)$.

Proof. The proof is essentially the same as the proof of Theorem 2.1 in [3]. □

The following result was established by Aubin (see [2]).

Theorem 3.2 (Aubin). *Let $\mathcal{B}_0, \mathcal{B}, \mathcal{B}_1$ be three Banach spaces where $\mathcal{B}_0, \mathcal{B}_1$ are reflexive. Suppose that \mathcal{B}_0 is continuously imbedded into \mathcal{B} , which is also continuously imbedded into \mathcal{B}_1 , and imbedding from \mathcal{B}_0 into \mathcal{B} is compact. For any given p_0, p_1 with $1 < p_0, p_1 < \infty$, let*

$$\mathcal{W} := \{v \mid v \in L^{p_0}(0, T; \mathcal{B}_0), v_t \in L^{p_1}(0, T; \mathcal{B}_1)\}.$$

Then the imbedding from \mathcal{W} into $L^{p_0}(0, T; \mathcal{B})$ is compact.

Let us introduce the following reflexive Banach spaces

$$X_T := \{\phi \mid \phi \in L^4(0, T; W^{11/8, 2}(\Omega)^3) \cap L^8(0, T; W^{1, 24/11}(\Omega)^3)\}$$

and

$$Y_T := \{\psi \mid \psi \in L^2(0, T; \mathcal{D}), \psi' \in L^2(0, T; H)\},$$

respectively, with norms

$$\|\phi\|_{X_T} := \|\phi\|_{L^4(0,T; W^{11/8,2}(\Omega)^3)} + \|\phi\|_{L^8(0,T; W^{1,24/11}(\Omega)^3)}$$

and

$$\|\psi\|_{Y_T} := \|\psi\|_{L^2(0,T; \mathcal{D})} + \|\psi'\|_{L^2(0,T; H)}.$$

Let us present some properties of X_T and Y_T . First note that [15, 16], [19, Corollary 4.4] and the requirements imposed on the domain Ω (see Subsection 1.2) yield

$$(18) \quad \mathcal{D} \hookrightarrow W^{2,2}(\Omega)^3,$$

which implies [13]

$$(19) \quad Y_T \hookrightarrow L^\infty(0, T; W^{1,2}(\Omega)^3).$$

Let $\phi \in Y_T$. Raising and integrating the interpolation inequality

$$\|\phi(t)\|_{W^{3/2,2}(\Omega)^3} \leq c \|\phi(t)\|_{W^{1,2}(\Omega)^3}^{1/2} \|\phi(t)\|_{W^{2,2}(\Omega)^3}^{1/2}$$

from 0 to T we get

$$(20) \quad \begin{aligned} \left(\int_0^T \|\phi(t)\|_{W^{3/2,2}(\Omega)^3}^4 dt \right)^{1/4} &\leq c \left(\int_0^T \|\phi(t)\|_{W^{2,2}(\Omega)^3}^2 \|\phi(t)\|_{W^{1,2}(\Omega)^3}^2 dt \right)^{1/4} \\ &\leq c \|\phi\|_{L^2(0,T; W^{2,2}(\Omega)^3)}^{1/2} \|\phi\|_{L^\infty(0,T; W^{1,2}(\Omega)^3)}^{1/2} \\ &\leq c \|\phi\|_{Y_T}, \end{aligned}$$

where $c = c(\Omega)$. Hence we have

$$(21) \quad Y_T \hookrightarrow L^4(0, T; W^{3/2,2}(\Omega)^3).$$

Using embeddings

$$W^{3/2,2}(\Omega)^3 \hookrightarrow W^{11/8,2}(\Omega)^3 \hookrightarrow L^2(\Omega)^3$$

and

$$W^{11/8,2}(\Omega)^3 \hookrightarrow W^{1,8/3}(\Omega)^3 \hookrightarrow L^{24}(\Omega)^3$$

and applying Theorem 3.2 we get the compact embedding

$$(22) \quad Y_T \hookrightarrow L^4(0, T; W^{11/8,2}(\Omega)^3) \hookrightarrow L^4(0, T; L^{24}(\Omega)^3).$$

Further, raising and integrating the interpolation inequality (cf. [1, Theorem 5.2])

$$\|\phi(t)\|_{W^{5/4,2}(\Omega)^3} \leq c \|\phi(t)\|_{W^{2,2}(\Omega)^3}^{1/4} \|\phi(t)\|_{W^{1,2}(\Omega)^3}^{3/4}$$

from 0 to T we get

$$(23) \quad \begin{aligned} \left(\int_0^T \|\phi(t)\|_{W^{5/4,2}(\Omega)^3}^8 dt \right)^{1/8} &\leq c \left(\int_0^T \|\phi(t)\|_{W^{2,2}(\Omega)^3}^2 \|\phi(t)\|_{W^{1,2}(\Omega)^3}^6 dt \right)^{1/8} \\ &\leq c \|\phi\|_{L^2(0,T; W^{2,2}(\Omega)^3)}^{1/4} \|\phi\|_{L^\infty(0,T; W^{1,2}(\Omega)^3)}^{3/4} \\ &\leq c \|\phi\|_{Y_T}, \end{aligned}$$

where $c = c(\Omega)$. Hence

$$(24) \quad Y_T \hookrightarrow L^8(0, T; W^{5/4,2}(\Omega)^3).$$

Note that

$$(25) \quad W^{5/4,2}(\Omega)^3 \hookrightarrow W^{9/8,2}(\Omega)^3 \hookrightarrow W^{1,24/11}(\Omega)^3.$$

Now 24 and 25 and Theorem 3.2 yield the compact embedding

$$(26) \quad Y_T \hookrightarrow L^8(0, T; W^{1,24/11}(\Omega)^3).$$

Finally, 22 and 26 imply the compact embedding

$$(27) \quad Y_T \hookrightarrow X_T.$$

4. MAIN RESULT

4.1. Statement of the result. The main result of the paper is the following

Theorem 4.1 (Main result). *There exists $T^* \in (0, T]$ and the uniquely determined function $\mathbf{u} \in L^2(0, T^*; \mathcal{D}) \cap L^\infty(0, T^*; V)$, $\mathbf{u}_t \in L^2(0, T^*; H)$, such that \mathbf{u} satisfies 12–13 for every $\mathbf{v} \in V$ and for almost every $t \in (0, T^*)$, where $\mathbf{f} \in L^2(0, T^*; H)$ and $\mathbf{u}_0 \in \mathcal{D}$.*

4.2. Proof of the main result.

4.2.1. Existence.

Remark 1. Setting $\mathbf{w} = \mathbf{u} - \mathbf{u}_0$ this amounts to solving the problem with the homogeneous initial condition

$$(28) \quad (\mathbf{w}_t, \mathbf{v}) + ((\mathbf{u}_0 + \mathbf{w}, \mathbf{v})) + b(\mathbf{u}_0 + \mathbf{w}, \mathbf{u}_0 + \mathbf{w}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})$$

$$(29) \quad \mathbf{w}(0) = \mathbf{0}$$

for every $\mathbf{v} \in V$ and for almost every $t \in (0, T)$, where $\mathbf{f} \in L^2(0, T; H)$ and $\mathbf{u}_0 \in \mathcal{D}$.

Denote by $B_R(T) \subset X_T$ the closed ball

$$(30) \quad B_R(T) := \{\varphi \in X_T; \|\varphi\|_{X_T} \leq R\}.$$

For arbitrary fixed $\tilde{\mathbf{w}} \in X_T$ we now consider the linear problem

$$(31) \quad (\mathbf{w}', \mathbf{v}) + ((\mathbf{w}, \mathbf{v})) = (\mathbf{f}, \mathbf{v}) - ((\mathbf{u}_0, \mathbf{v})) - b(\mathbf{u}_0, \mathbf{u}_0, \mathbf{v}) - b(\mathbf{u}_0, \tilde{\mathbf{w}}, \mathbf{v}) \\ - b(\tilde{\mathbf{w}}, \mathbf{u}_0, \mathbf{v}) - b(\tilde{\mathbf{w}}, \tilde{\mathbf{w}}, \mathbf{v})$$

$$(32) \quad \mathbf{w}(0) = \mathbf{0}$$

for every $\mathbf{v} \in V$ and for almost every $t \in (0, T)$.

Definition 4.2. Let $\mathcal{F} : X_T \rightarrow L^2(0, T; H)$ be an operator such that

$$(33) \quad (\mathcal{F}(\phi), \mathbf{v}) = (\mathbf{f}, \mathbf{v}) - ((\mathbf{u}_0, \mathbf{v})) - b(\mathbf{u}_0, \mathbf{u}_0, \mathbf{v}) \\ - b(\mathbf{u}_0, \phi, \mathbf{v}) - b(\phi, \mathbf{u}_0, \mathbf{v}) - b(\phi, \phi, \mathbf{v})$$

for every $\mathbf{v} \in V$ and for almost every $t \in (0, T)$.

From Theorem 3.1 we deduce that for arbitrary fixed $\tilde{\mathbf{w}} \in X_T$ there exists $\mathbf{w} \in L^2(0, T; \mathcal{D}) \cap L^\infty(0, T; V)$, $\mathbf{w}' \in L^2(0, T; H)$, $\mathbf{w}(0) = \mathbf{0}$ and

$$(34) \quad \|\mathbf{w}\|_{X_T} \leq c \|\mathcal{F}(\tilde{\mathbf{w}})\|_{L^2(0, T; H)}.$$

We now prove the following

Lemma 4.3. \mathcal{F} is a continuous operator from X_T into $L^2(0, T; H)$ and for all $R > 0$, $T > 0$, and for all $\tilde{\mathbf{w}} \in B_R(T)$ we have

$$(35) \quad \|\mathcal{F}(\tilde{\mathbf{w}})\|_{L^2(0, T; H)} \leq C_0(T) + C_1(T^{1/8}R^2 + T^{1/4}R),$$

where $C_0(T) \rightarrow 0_+$ for $T \rightarrow 0_+$ and C_1 is independent of T .

Proof. Obviously, there exists $C_0(T) > 0$ such that

$$(36) \quad \|(\mathbf{f}, \cdot) - ((\mathbf{u}_0, \cdot)) - b(\mathbf{u}_0, \mathbf{u}_0, \cdot)\|_{L^2(0,T;H)} \leq C_0(T),$$

$C_0(T) \rightarrow 0_+$ for $T \rightarrow 0_+$.

Using the interpolation inequality one obtains

$$(37) \quad \begin{aligned} \|b(\tilde{\mathbf{w}}, \mathbf{u}_0, \cdot)\|_{L^2(0,T;H)} &\leq \left(\int_0^T \|\mathbf{u}_0\|_{W^{1,4}(\Omega)^3}^2 \|\tilde{\mathbf{w}}\|_{L^4(\Omega)^3}^2 dt \right)^{1/2} \\ &\leq \|\mathbf{u}_0\|_{W^{1,4}(\Omega)^3} \|\tilde{\mathbf{w}}\|_{L^2(0,T;L^4(\Omega)^3)} \\ &\leq T^{1/4} \|\mathbf{u}_0\|_{W^{1,4}(\Omega)^3} \|\tilde{\mathbf{w}}\|_{L^4(0,T;L^4(\Omega)^3)} \\ &\leq c T^{1/4} \|\mathbf{u}_0\|_{\mathcal{D}} \|\tilde{\mathbf{w}}\|_{X_T}. \end{aligned}$$

Similarly, we obtain the inequalities

$$(38) \quad \begin{aligned} \|b(\mathbf{u}_0, \tilde{\mathbf{w}}, \cdot)\|_{L^2(0,T;H)} &\leq \left(\int_0^T \|\mathbf{u}_0\|_{L^\infty(\Omega)^3}^2 \|\tilde{\mathbf{w}}\|_{W^{1,2}(\Omega)^3}^2 dt \right)^{1/2} \\ &\leq \|\mathbf{u}_0\|_{L^\infty(\Omega)^3} \|\tilde{\mathbf{w}}\|_{L^2(0,T;W^{1,2}(\Omega)^3)} \\ &\leq T^{1/4} \|\mathbf{u}_0\|_{L^\infty(\Omega)^3} \|\tilde{\mathbf{w}}\|_{L^4(0,T;W^{1,2}(\Omega)^3)} \\ &\leq c T^{1/4} \|\mathbf{u}_0\|_{\mathcal{D}} \|\tilde{\mathbf{w}}\|_{X_T} \end{aligned}$$

and

$$(39) \quad \begin{aligned} \|b(\tilde{\mathbf{w}}, \tilde{\mathbf{w}}, \cdot)\|_{L^2(0,T;H)} &\leq \left(\int_0^T \|\tilde{\mathbf{w}}\|_{L^{24}(\Omega)^3}^2 \|\tilde{\mathbf{w}}\|_{W^{1,24/11}(\Omega)^3}^2 dt \right)^{1/2} \\ &\leq T^{1/8} \|\tilde{\mathbf{w}}\|_{L^4(0,T;L^{24}(\Omega)^3)} \|\tilde{\mathbf{w}}\|_{L^8(0,T;W^{1,24/11}(\Omega)^3)} \\ &\leq c T^{1/8} \|\tilde{\mathbf{w}}\|_{X_T}^2. \end{aligned}$$

The inequalities 36–39 yield $\mathcal{F}(\tilde{\mathbf{w}}) \in L^2(0, T; H)$ and the inequality 35 holds.

Let $\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2 \in X_T$ and $\mathbf{z} = \tilde{\mathbf{w}}_2 - \tilde{\mathbf{w}}_1$. Then

$$(40) \quad \begin{aligned} \|\mathcal{F}(\tilde{\mathbf{w}}_2) - \mathcal{F}(\tilde{\mathbf{w}}_1)\|_{L^2(0,T;H)} &\leq \|\mathbf{b}(\mathbf{z}, \mathbf{u}_0, \cdot)\|_{L^2(0,T;H)} \\ &\quad + \|\mathbf{b}(\mathbf{u}_0, \mathbf{z}, \cdot)\|_{L^2(0,T;H)} + \|\mathbf{b}(\tilde{\mathbf{w}}_2, \mathbf{z}, \cdot)\|_{L^2(0,T;H)} + \|\mathbf{b}(\mathbf{z}, \tilde{\mathbf{w}}_1, \cdot)\|_{L^2(0,T;H)} \end{aligned}$$

and

$$(41) \quad \begin{aligned} \|b(\tilde{\mathbf{w}}_2, \mathbf{z}, \cdot)\|_{L^2(0,T;H)} &\leq \left(\int_0^T \|\tilde{\mathbf{w}}_2\|_{L^{24}(\Omega)^3}^2 \|\mathbf{z}\|_{W^{1,24/11}(\Omega)^3}^2 dt \right)^{1/2} \\ &\leq \|\tilde{\mathbf{w}}_2\|_{L^4(0,T;L^{24}(\Omega)^3)} \|\mathbf{z}\|_{L^4(0,T;W^{1,24/11}(\Omega)^3)} \\ &\leq c \|\tilde{\mathbf{w}}_2\|_{X_T} \|\mathbf{z}\|_{X_T}. \end{aligned}$$

Similarly

$$(42) \quad \begin{aligned} \|b(\mathbf{z}, \tilde{\mathbf{w}}_1, \cdot)\|_{L^2(0,T;H)} &\leq \left(\int_0^T \|\mathbf{z}\|_{L^{24}(\Omega)^3}^2 \|\tilde{\mathbf{w}}_1\|_{W^{1,24/11}(\Omega)^3}^2 dt \right)^{1/2} \\ &\leq c \|\mathbf{z}\|_{X_T} \|\tilde{\mathbf{w}}_1\|_{X_T}. \end{aligned}$$

Inequalities 41–42 and 37–38 imply that \mathcal{F} is a continuous operator from X_T into $L^2(0, T; H)$. \square

The proof of the main result is based on the Brouwer fixed point theorem. Let the operator \mathcal{A} be defined as follows. Given a function $\tilde{w} \in X_T$, consider the linear problem

$$(43) \quad (\mathbf{w}', \mathbf{v}) + ((\mathbf{w}, \mathbf{v})) = (\mathcal{F}(\tilde{w}), \mathbf{v})$$

$$(44) \quad \mathbf{w}(0) = \mathbf{0}$$

for every $\mathbf{v} \in V$ and for almost every $t \in (0, T)$, where \mathcal{F} is defined by Definition 4.2. Theorem 3.1 and Lemma 4.3 ensure that the linear problem 43–44 has a unique solution $\mathbf{w} \in Y_T$. Define $\mathcal{A} : X_T \rightarrow Y_T$ by setting $\mathcal{A}(\tilde{w}) = \mathbf{w}$. Clearly, the inequality 17 and Lemma 4.3 imply that \mathcal{A} is a continuous operator from X_T into Y_T . For all $\tilde{w} \in B_R(T)$, taking 17 and 35 together, we deduce

$$\|\mathcal{A}(\tilde{w})\|_{X_T} \leq c_1 \|\mathcal{A}(\tilde{w})\|_{Y_T} \leq c_2 \|\mathcal{F}(\tilde{w})\|_{L^2(0, T^*; H)} \leq c_3(T) + c_4(T^{1/8}R^2 + T^{1/4}R),$$

where $c_3(T) \rightarrow 0$ for $T \rightarrow 0$ and c_1, c_2 and c_4 do not depend on T . Hence for $T = T^*$, $T^* > 0$ sufficiently small, and for a sufficiently large R , \mathcal{A} maps $B_R(T^*)$ into itself. Since \mathcal{A} is a continuous operator from X_T into Y_T and $Y_T \hookrightarrow X_T$, \mathcal{A} is totally continuous operator from X_{T^*} into X_{T^*} , where X_{T^*} is a reflexive Banach space. Therefore there exists a fixed point $\mathbf{w} \in B_R(T^*)$ such that $\mathcal{A}(\mathbf{w}) = \mathbf{w}$ in X_{T^*} .

4.2.2. *Uniqueness.* Suppose that there are two solutions $\mathbf{u}_1, \mathbf{u}_2 \in Y_{T^*}$ of 12–13 on $(0, T^*)$. Denote $\mathbf{z} = \mathbf{u}_1 - \mathbf{u}_2$ then

$$(45) \quad (\mathbf{z}_t, \mathbf{v}) + ((\mathbf{z}, \mathbf{v})) + b(\mathbf{z}, \mathbf{u}_2, \mathbf{v}) + b(\mathbf{u}_1, \mathbf{z}, \mathbf{v}) = 0$$

holds for all $\mathbf{v} \in V$ and almost every $t \in (0, T)$ and $\mathbf{z}(0) = \mathbf{0}$. Hence

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{z}(t)\|_H^2 + \|\mathbf{z}(t)\|_V^2 &\leq |b(\mathbf{u}_1(t), \mathbf{z}(t), \mathbf{z}(t))| + |b(\mathbf{z}(t), \mathbf{u}_2(t), \mathbf{z}(t))| \\ &\leq \|\mathbf{u}_1(t)\|_{L^4(\Omega)^3} \|\nabla \mathbf{z}(t)\|_{L^2(\Omega)^3} \|\mathbf{z}(t)\|_{L^4(\Omega)^3} \\ &\quad + \|\mathbf{z}(t)\|_{L^4(\Omega)^3}^2 \|\nabla \mathbf{u}_2(t)\|_{L^2(\Omega)^3}. \end{aligned}$$

Using the interpolation inequality

$$\|\mathbf{z}(t)\|_{L^4(\Omega)^3} \leq c \|\mathbf{z}(t)\|_V^{3/4} \|\mathbf{z}(t)\|_{L^2(\Omega)^3}^{1/4}$$

we get

$$(46) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{z}(t)\|_H^2 + \|\mathbf{z}(t)\|_V^2 &\leq c_1 \|\mathbf{u}_1(t)\|_{L^4(\Omega)^3} \|\mathbf{z}(t)\|_V^{7/4} \|\mathbf{z}(t)\|_{L^2(\Omega)^3}^{1/4} \\ &\quad + c_2 \|\mathbf{z}(t)\|_V^{3/2} \|\mathbf{z}(t)\|_{L^2(\Omega)^3}^{1/2} \|\mathbf{u}_2(t)\|_{W^{1,2}(\Omega)^3}. \end{aligned}$$

Using Young's inequality we deduce

$$(47) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{z}(t)\|_H^2 + \|\mathbf{z}(t)\|_V^2 &\leq \delta \|\mathbf{z}(t)\|_V^2 \\ &\quad + c_\delta \|\mathbf{z}(t)\|_{L^2(\Omega)^3}^2 \left(\|\mathbf{u}_1(t)\|_{L^4(\Omega)^3}^8 + \|\mathbf{u}_2(t)\|_{W^{1,2}(\Omega)^3}^4 \right), \end{aligned}$$

where $\delta > 0$ can be chosen arbitrarily small and therefore

$$(48) \quad \frac{d}{dt} \|\mathbf{z}(t)\|_H^2 \leq 2c_\delta \|\mathbf{z}(t)\|_H^2 \left(\|\mathbf{u}_1(t)\|_{L^4(\Omega)^3}^8 + \|\mathbf{u}_2(t)\|_{W^{1,2}(\Omega)^3}^4 \right).$$

Hence, we have the differential inequality

$$y'(t) \leq \theta(t)y(t),$$

where

$$y(t) = \|\mathbf{z}(t)\|_H^2 \quad \text{and} \quad \theta(t) = 2c_\delta \left(\|\mathbf{u}_1(t)\|_{L^4(\Omega)^3}^8 + \|\mathbf{u}_2(t)\|_{W^{1,2}(\Omega)^3}^4 \right) \in L^1((0, T)),$$

from which we obtain, using the technique of Gronwall's lemma,

$$\frac{d}{dt} \left(y(t) \exp \left(- \int_0^t \theta(s) \, ds \right) \right) \leq 0$$

and

$$y(t) \leq y(0) \exp \left(\int_0^t \theta(s) \, ds \right).$$

Therefore

$$\|\mathbf{z}(t)\|_H^2 \leq \|\mathbf{z}(0)\|_H^2 \exp \left(\int_0^t 2c_\delta \left(\|\mathbf{u}_1(s)\|_{L^4(\Omega)^3}^8 + \|\mathbf{u}_2(s)\|_{W^{1,2}(\Omega)^3}^4 \right) \, ds \right)$$

for all $t \in (0, T)$. Now the uniqueness follows from the fact that $\mathbf{z}(0) = \mathbf{0}$.

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